High order expansion of matrix models and enumeration of maps.

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February 2, 2008

Abstract Perturbation of the **GUE** are known in physics to be related to enumeration of graphs on surfaces. Following [7] and [8], we investigate this idea and show that for a small convex perturbation, we can perform a genus expansion: the moments of the empirical measure can be developed into a series whose g-th term is a generating function of graphs on a surface of genus g.

1 Introduction

Wick's calculus allows to easily compute any moments of Gaussian variables and gives them a combinatorial interpretation since the p-th moment of a Gaussian can be seen as the number of partitions in pairs of $[|1, p|]$. This fact can be used to find moments of the **GUE**, the Gaussian unitary model. Let μ^N be the law on $\mathcal{H}_N(\mathbb{C})^m$ the set of m-tuple $A_1, \cdots, A_m$ of $N \times N$ hermitian matrices such that $\Re e A_i(kl), k < l$, $\Im m A_i(kl), k < l$, $2^{-\frac{1}{2}} A_i(kk)$ is a family of independent real Gaussian of variance $(2N)^{-1}$ or more directly

$$\mu^N(d\mathbf{A}) = \frac{1}{Z^N} e^{-\frac{N}{2} \operatorname{tr}(\sum_{i=1}^m A_i^2)} d^N \mathbf{A}$$

with $d^N \mathbf{A}$ the Lebesgue measure on $\mathcal{H}_N(\mathbb{C})^m = (\mathbb{R}^{N^2})^m$ and Z^N a constant of normalization.

For a edge-colored graph on an orientated surface we say that a vertex is of type $q = X_{i_1} \cdots X_{i_p}$ for a monomial q if this vertex is of valence p and when we look at the half-edges going out of it, starting from a distinguished

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one and going in the clockwise order the first half-edge is of color i_1, the second of color $i_2,\ldots$, the p-th of color i_p. A graph on a surface is a map if it is connected and its faces are homeomorphic to discs (see section 3 for a precise definition of these notions). Then, a simple computation (a simple generalization of [10]) using Wick's calculus shows that for all non commutative monomials, $X_{i_1} \cdots X_{i_p}$,

$$\mu^N(\frac{1}{N}\operatorname{tr}(A_{i_1} \cdots A_{i_p})) = \sum_{g \in \mathbb{N}} \frac{1}{N^{2g}} \mathcal{M}_g(X_{i_1} \cdots X_{i_p})$$

where $\mathcal{M}_g(X_{i_1} \cdots X_{i_p})$ is the number up to isomorphism of maps with colored edges on a surface of genus g with one vertex of type $X_{i_1} \cdots X_{i_p}$. Besides, one can use Euler's formula to show that the sum in the right hand side is always finite.

Is such an interpretation can be generalized beyond the Gaussian case? Take a potential $V(X_1, \cdots, X_m) = \sum_i t_i q_i$ with complex $t_1, \cdots, t_n$ and non-commutative monomials q_i. We are interested in the following perturbation of the **GUE**

$$\mu_V^N(dA_1, \cdots, dA_m) = \frac{1}{Z_V^N} e^{-N \operatorname{tr}(V(A_1, \cdots, A_m))} d\mu^N(A_1, \cdots, A_m) \qquad (1)$$

where Z_V^N is the normalizing constant so that μ_V^N is a probability measure. The derivatives of the moments of this model in $\mathbf{t} = 0$ are exactly moments of the **GUE** and thus can be computed using Wick's calculus and the limit can be formally expressed as a generating function of graphs. For example, for $V = tX^4$, we can obtain the following expansion for the free energy

$$\frac{1}{N^2} \log Z_{tX^4}^N = \sum_{g \in \mathbb{N}} \frac{1}{N^{2g}} \sum_{k \in \mathbb{N}} \frac{(-t)^k}{k!} \mathcal{C}_g^k \qquad (2)$$

with $\mathcal{C}_g^k$ the number up to isomorphism of connected graphs on a surface of genus g with k vertex of valence 4 and such that faces are homeomorphic to discs (the so-called maps). Note that we have to be careful since the right hand side of (2) is divergent for $t \neq 0$. Thus, this equality is purely formal but we will be able to give it a precise mathematical meaning (at least, Wick's calculus show that the derivatives of both sides are equal in $t = 0$).

Such genus expansion has been used for long in physics and one may wonder for which potentials we can prove it. In the one matrix case, the problem has been solved in [4] using Riemann-Hilbert methods. This case has also

been studied a lot in physics, see for example the review [3]. The multi-matrix case is much complicated since the technique of orthogonal polynomials can not be generalized. Although, there is a large literature in physics (see [5], [2] for example) on some specific cases such as the Ising model on random graphs i.e. a potential of the form $V = V_1(A) + V_2(B) + cAB$. In mathematics, this problem has also been studied with a completely different approach, namely large deviation in [6], which gives the first asymptotic of models closed to the Ising model.

In this paper we will try to avoid the use of orthogonal polynomials and our main tool will be the so-called Schynger-Dyson's equation. Besides, we will concentrate on the combinatorial interpretation of the limit. Our study follows the two papers [7] and [8] which respectively studied the first order asymptotic and a central limit theorem for these models.

We will always assume

1. the perturbation is small, we will restrict ourselves to small coefficients t_i in V. Note that we can not get rid of this condition as the generating functions of combinatorial objects that appear have arbitrary small radius of convergence.

2. the potential $V + \frac{1}{2}\sum_i X_i^2$ is "uniformly" convex: there exists $c > 0$ such that for all N in $\mathbb{N}$,

$$\varphi_V^N : \begin{array}{ccc} \mathcal{H}_N(\mathbb{C})^m & \longrightarrow & \mathbb{C} \\ (X_1, \cdots, X_m) & \longrightarrow & \mathrm{tr}(V(X_1, \cdots, X_m) + \frac{1-c}{2}\sum_{i=1}^m X_i^2) \end{array}$$

is a real and convex function. If V satisfy this condition, we will say that V is c-convex.

Thus, for $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with $\mathbf{t} = (t_1, \cdots, t_n)$ complex numbers and q_i non-commutative monomials we define

$$B_{\eta, c} = \{\mathbf{t} \in \mathbb{C}^n | \|\mathbf{t}\| = \max_i |t_i| \leqslant \eta, V \text{ is } c\text{-convex}\}.$$

An example of c-convex function is

$$V(\mathbf{X}) = \sum_i P_i(\sum_j \alpha_{ij} X_j) + \sum_{k\ell} \beta_{k\ell} X_k X_\ell$$

with real and convex polynomials P_i, real α_{ij}, β_{kl} and for all l, $\sum |\beta_{kl}| < 1-c$.

The main result of this paper is

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Theorem 1.1 *Let $V_{\mathbf{t}} = \sum_{i=1}^{n} t_i q_i$, and $c > 0$, for all $g \in \mathbb{N}$, there exists $\eta_g > 0$ such that for all $\mathbf{t}$ in $B_{\eta_g,c}$, the free energy has the following expansion*

$$F_{V_{\mathbf{t}}}^N := \frac{1}{N^2} \log Z_{V_{\mathbf{t}}}^N = F^0(\mathbf{t}) + \frac{1}{N^2} F^1(\mathbf{t}) + \cdots + \frac{1}{N^{2g}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

with F^g the generating function for maps of genus g associated with V:

$$F^g(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{N}^n} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{C}_g^{\mathbf{k}}(P)$$

where $\mathbf{k}! = \prod_i k_i!$, $(-\mathbf{t})^{\mathbf{k}} = \prod_i (-t_i)^{k_i}$ and $\mathcal{C}_g^{\mathbf{k}}$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

In addition to the free energy, we are interested in more general observables. For example we would like to know the asymptotic of the non-commutative moments of our measure $E_{\mu_{V_{\mathbf{t}}}^N}[\frac{1}{N}\operatorname{tr}(P)]$ for a non-commutative polynomial P. Such moments appear as derivatives of the free energy since

$$-E_{\mu_{V_{\mathbf{t}}}^N}[\frac{1}{N}\operatorname{tr}(P)] = \frac{\partial}{\partial u}\Big|_{u=0} F_{V_{\mathbf{t}}+uP}^N.$$

Theorem 1.2 *With the same hypothesis than in the previous theorem, for all for all $g \in \mathbb{N}$, there exists $\eta > 0$, such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all monomials P*

$$E_{\mu_{V_{\mathbf{t}}}^N}[\frac{1}{N}\operatorname{tr}(P)] = \mathcal{C}_{\mathbf{t}}^0(P) + \cdots + \frac{1}{N^{2g}}\mathcal{C}_{\mathbf{t}}^g(P) + o(\frac{1}{N^{2g}})$$

with $\mathcal{C}_g$ the generating function maps of genus g with some fixed vertices:

$$\mathcal{C}_{\mathbf{t}}^g(P) = \sum_{\mathbf{k} \in \mathbb{N}^n} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{C}_g^{\mathbf{k}}(P)$$

where $\mathcal{C}_g^{\mathbf{k}}(P)$ is the number of maps on a surface of genus g with k_i vertices of type q_i and one of type P.

In fact, we will be able to find the asymptotic of much more observable such as the higher derivatives of the free energy. Indeed, we show that we can derivate term by term the expansion of Theorem 1.1. Let us introduce for $\mathbf{j} = (j_1, \cdots, j_n) \in \mathbb{N}^n$, the operator of derivation

$$\mathcal{D}_{\mathbf{j}} = \frac{\partial^{\sum_i j_i}}{\partial t_1^{j_1} \cdots \partial t_n^{j_n}}.$$

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Theorem 1.3 *With the same hypothesis than in the previous theorem, for all* $\mathbf{j} = (j_1, \cdots, j_n) \in \mathbb{N}^n$, *for all* $g \in \mathbb{N}$, *there exists* $\eta > 0$ *such that for all* $\mathbf{t}$ *in* $B_{\eta,c}$,

$$\mathcal{D}_{\mathbf{j}} F_{V_{\mathbf{t}}}^N = \mathcal{D}_{\mathbf{j}} F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}} \mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}}).$$

Besides, $\mathcal{D}_{\mathbf{j}} F^g$ *is the generating function maps of genus* g *with some fixed vertices:*

$$\mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{N}^n} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{C}_g^{\mathbf{k}+\mathbf{j}}.$$

In the next section, we will define some useful notation when one has to deal with non-commutative probability theory and we will recall the main result of [7]. Next, we will begin to look for recursive relations between the asymptotic of the non-commutative moments of our model. This will lead us to study some combinatorial objects in section 4 whose generating function satisfy these relations. In the two next sections, we will prove the equality of these moments and these enumerating functions before proving our main results. Finally the last section will be devoted to the proof of Theorem 1.3.

2   Notations and reminder

We denote by $\mathbb{C}\langle X_1, \cdots, X_m \rangle$ the set of complex polynomials on the non-commutative unknown $X_1, \ldots, X_m$ i.e. the complex linear combination of monomials which are simply the set of finite words on $X_1, \ldots, X_m$. Monomials must be thought as non-commutative moments. Let $*$ denotes the linear involution on $\mathbb{C}\langle X_1, \cdots, X_m \rangle$ such that for all complex z and all monomials

$$\left(z X_{i_1} \ldots X_{i_p} \right)^* = \overline{z} X_{i_p} \ldots X_{i_1}.$$

We will say that a polynomial P is self-adjoint if $P = P^*$. We will denote $\mathbb{C}\langle X_1, \cdots, X_m \rangle^*$ the dual of $\mathbb{C}\langle X_1, \cdots, X_m \rangle$. An element of $\mathbb{C}\langle X_1, \cdots, X_m \rangle^*$ has a compact support if there exists $R > 0$ such that for all monomial $|\tau(X_{i_1} \ldots X_{i_p})| \leqslant R^p$. By analogy with the one variable case, the infimum of the R's which satisfy this inequality for all monomials will be called the radius of the support of τ.

For a polynomial P and a monomial q, we define $\lambda_q(P)$ as the coefficient of q in the decomposition of P. For $M > 0$, we define the norm $\|.\|_M$ on polynomials:

$$\|P\|_M = \sum_{l \in \mathbb{N}} \sum_{\substack{q \text{ monomial} \\ deg q = l}} |\lambda_q(P)| M^l.$$

This norm $\|.\|_M$ is an algebra norm, i.e. for all polynomials P, Q,

$$\|PQ\|_M \leqslant \|P\|_M \|Q\|_M.$$

Note that an element τ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ has support of radius less than R if and only if for all polynomials P,

$$|\tau(P)| \leqslant \|P\|_R.$$

We extend $\|.\|_M$ on $\mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle$ by defining this norm on the decomposition in monomials:

$$\|\sum \lambda_{q_1,q_2} q_1 \otimes q_2\|_M = \sum |\lambda_{q_1,q_2}| M^{\deg q_1 + \deg q_2}$$

with this definition for all polynomials P, Q, $\|P \otimes Q\|_M = \|P\|_M \|Q\|_M$.

We define the non-commutative derivatives ∂_i from $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 2}$ for $1 \leqslant i \leqslant m$ by the Leibniz rule

$$\partial_i PQ = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q$$

and $\partial_i X_j = \mathbb{1}_{i=j} 1 \otimes 1$. For a monomial P, we will often use the convenient expression

$$\partial_i P = \sum_{P=RX_iS} R \otimes S$$

where the sum runs over all possible monomials R, S so that P decomposes into RX_iS. We also define another operator of derivation on polynomials, the cyclic derivative D_i which is linear and such that for all monomials:

$$D_i P = \sum_{P=RX_iS} SR.$$

Alternatively, D can be defined as $m \circ \partial$ where $m(A \otimes B) = BA$. We will see that these two operators appear naturally when we derivate products of matrices and they both possesses a nice combinatorial interpretation. An important fact we will use later is that for all $M' > M$, both ∂_i and D_i are continuous from $(\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_{M'})$ to $(\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_M)$. For example for a monomial q,

$$\frac{\|D_i q\|_M}{\|q\|_{M'}} = \deg q \frac{M^{\deg q - 1}}{M'^{\deg q}} = M^{-1} \deg q (\frac{M}{M'})^{\deg q}$$

which is bounded.

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Our main object of study is the law $\mu_{V_{\mathbf{t}}}^N$ on $\mathcal{H}_N(\mathbb{C})^m$

$$\mu_{V_{\mathbf{t}}}^N(dA_1, \cdots, dA_m) = \frac{1}{Z_{V_{\mathbf{t}}}^N} e^{-N \operatorname{tr}(V_{\mathbf{t}}(A_1, \cdots, A_m))} d\mu^N(A_1, \cdots, A_m)$$

and we are particulary interested by the behavior of the random variable

$$\hat{\mu}^N : \begin{array}{ccc} \mathbb{C}\langle X_1, \cdots, X_m \rangle & \longrightarrow & \mathbb{C} \\ P & \longrightarrow & \frac{1}{N} \operatorname{tr}(P(A_1, \cdots, A_m)) \end{array}$$

and its mean:

$$\overline{\mu}_{\mathbf{t}}^N : \begin{array}{ccc} \mathbb{C}\langle X_1, \cdots, X_m \rangle & \longrightarrow & \mathbb{C} \\ P & \longrightarrow & E_{\mu_{V_{\mathbf{t}}}^N}[\frac{1}{N} \operatorname{tr}(P(A_1, \cdots, A_m))] \end{array}$$

We can now state precisely the main result of [7], we will overuse it in the next sections. For any c, there exists $\eta > 0$ such that for all $\mathbf{t} \in B_{\eta,c}$, for all polynomials P, $\hat{\mu}^N(P)$ goes when N goes to $+\infty$, almost surely and in expectation towards $\mu_{\mathbf{t}}(P)$ with $\mu_{\mathbf{t}}$ a solution of the Schwinger-Dyson equation

$$\mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_i P) = \mu_{\mathbf{t}}((X_i + D_i V_{\mathbf{t}})P) \quad \forall P \in \mathbb{C}\langle X_1, \cdots, X_m \rangle, \forall i \in \{1, \cdots, m\}.$$

Besides, this soluion has a bounded support, there exists R such that for all monomial $X_{i_1} \cdots X_{i_p}$

$$|\mu_{\mathbf{t}}(X_{i_1} \cdots X_{i_p})| \leqslant R^p. \tag{3}$$

and $\mu_{\mathbf{t}}$ can be seen as a generating function of planar maps, for all polynomials P,

$$\mu_{\mathbf{t}}(P) = \sum_{k_1, \cdots, k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{k_1, \cdots, k_n}(P)$$

where $\mathcal{M}^{k_1, \cdots, k_n}(P)$ is the number of planar connected graphs with k_i vertices of type q_i and one of type P. For the rest of the paper we will work in this domain $B_{\eta,c}$ where the convergence holds. In order to shorten a little the notations the subscript $\mathbf{t}$ will be most of the time implicit, for example we will often write μ instead of $\mu_{\mathbf{t}}$, V instead of $V_{\mathbf{t}}$, $\overline{\mu}^N$ instead of $\overline{\mu}_{\mathbf{t}}^N$ ...

3 First order observable

The starting point is a relation already used in [7] for the matrix model when N is fixed: for all polynomial P, for all i,

$$E[\hat{\mu}^N((X_i + D_i V)P)] = E[(\hat{\mu}^N \otimes \hat{\mu}^N)(\partial_i P)].$$

We will give the proof of a generalization of this equality later. Using this equality and some concentration inequalities we were able to prove that for $\mathbf{t}$ in $B_{\eta,c}$ for a well chosen η, for all polynomial P, $E[\hat{\mu}^N(P)]$ was converging towards $\mu(P)$ with μ the unique solution of the Swinger-Dyson's equation $\mathbf{SD[V]}$:

$$\mu((X_i + D_i V)P)) = (\mu \otimes \mu)(\partial_i P). \tag{4}$$

In order to find the next asymptotic, we study the difference between the equation for finite N and the limit equation, if $\nu^N = N^2(\overline{\mu}^N - \mu)$, we obtain by substracting the two equations:

$$\nu^N((X_i + D_i V)P) - (I \otimes \mu + \mu \otimes I)\partial_i P) = N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(\partial_i P)] \tag{5}$$

Here, an important operator shows up in the left hand side. Let $P = X_{i_1} \cdots X_{i_p}$ be a monomial and define the following operators:

$$\Xi_1 P = \frac{1}{p} \sum_i D_i V D_i P$$

$$\Xi_2 P = \frac{1}{p} \sum_i (I \otimes \mu + \mu \otimes I)\partial_i D_i P.$$

We extend them by linearity on $\mathbb{C}\langle X_1, \cdots, X_m \rangle$ and we define $\Xi_0 = I - \Xi_2$ and $\Xi = \Xi_0 + \Xi_1$. These operators were introduced in [8] to obtain a central limit theorem for the matrix model. We also define the operator of division by the degree i.e. the linear operator $P \to \overline{P}$ such that for all monomial $P = X_{i_1} \cdots X_{i_p}$, $\overline{P} = \frac{1}{p} P$ and $\overline{1} = 0$. These operators allow us to state the relation for the first correction (5) in a simpler form:

$$\nu^N(\Xi P) = N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(\partial_i D_i \overline{P})]. \tag{6}$$

Then, the strategy is simple, we only have to understand the asymptotic of $N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(R \otimes S)]$ and then "invert" Ξ. The first order asymptotic of $N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(R \otimes S)]$ is easy to compute using [8] as it was shown that $N(\hat{\mu}^N - \mu)$ converges in law towards a Gaussian when N goes to infinity. The main issue is that when we try to investigate the next asymptotic, terms of type $N^3 E[(\hat{\mu}^N - \mu)^{\otimes 3}(R \otimes S \otimes T)]$ will appear and at their turn they will create terms of greater complexity. That's why we are interested more generally in all the $N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)]$'s and we will eventually find their full asymptotic. First remark that according to [8], for all P, $N(\hat{\mu}^N(P) - \mu(P))$ converges to a gaussian variable and this convergence occurs in moments (see Corollary 4.8 in [8]). Thus $N^\ell E[(\hat{\mu}^N -$

$\mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)]$ has a finite limit when N goes to infinity and this limit is 0 if ℓ is odd. But we need a more precise result which state that this convergence is uniform for all monomials P of reasonable degree.

Lemma 3.1 *For all $\ell \in \mathbb{N}^*$, $\alpha > 0$ there exists $C, \eta, M_0 > 0$, such that for all $\mathbf{t} \in B_{\eta,c}$, $M > M_0$ and all polynomials $P_1, \ldots, P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$,*

$$|E[N^\ell(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)]| \le C\|P_1\|_M \cdots \|P_\ell\|_M.$$

Proof.

First using Hölder's inequality, write

$$E[N^\ell(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)] \le \prod_{r=1}^\ell E[|N(\hat{\mu}^N - \mu)(P_r)|^\ell]^{\frac{1}{\ell}}.$$

Thus we only have to prove the claim if the P_r are equals. Then we substract the mean

$$E[|N(\hat{\mu}^N - \mu)(P)|^\ell] \le 2^\ell \{E[|N(\hat{\mu}^N(P) - \overline{\mu}^N(P)|^\ell] + |N(\overline{\mu}^N(P) - \mu(P))|^\ell\} \tag{7}$$

Proposition 3.1 in [8] state a rate of convergence of $\overline{\mu}^N(P)$ to $\mu(P)$: there exists $C, M_0 > 0$ such that for $M > M_0$, for all polynomials P of degree less than $\varepsilon N^{\frac{2}{3}}$,

$$|N(\overline{\mu}^N(P) - \mu(P))| \le C\frac{\|P\|_M}{N}. \tag{8}$$

Thus in inequality (7) above, we only have to control the first term. As we restrict ourself on the domain where the support of μ is uniformly bounded by R, according to (3) we get that for all polynomials of degree less than $N^{\frac{1}{2}}$,

$$|\overline{\mu}^N(P)| \le |\mu(P)| + C\frac{\|P\|_M}{N} \le C'\|P\|_M$$

for M larger than M_0, R. We split the first term in two, let $\hat{\mu}_-^N = \hat{\mu}^N \mathbb{1}_{\|\mathbf{A}\| \le M}$ and $\hat{\mu}_+^N = \hat{\mu}^N \mathbb{1}_{\|\mathbf{A}\| > M}$ with $\|.\|$ the operator-norm. By Jensen's inequality

$$E[|N(\hat{\mu}_+^N(P) - E[\hat{\mu}_+^N(P)]|^\ell] \le (2N)^\ell \overline{\mu}^N((PP^*)^\ell)^{\frac{1}{2}} \mathbb{P}(\|\mathbf{A}\| > M)^{\frac{1}{2}}$$

Now according to Lemma 2.2 in [8], there exists $\alpha > 0$ such that if M is sufficiently large, $\mathbb{P}(\|\mathbf{A}\| > M) \le e^{-\alpha MN}$. Thus

$$E[|N(\hat{\mu}_+^N(P) - E[\hat{\mu}_+^N(P)]|^\ell] \le C\|(PP^*)^\ell\|_M^{\frac{1}{2}} \le C\|P\|_M^\ell \tag{9}$$

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Observe that $\hat{\mu}^N(P)$ is lipschitz on $\{\|\mathbf{A}\| \leqslant M\}$ with lipschitz constant

$$
\sup_{\{\|\mathbf{A}\| \leqslant M\}} (\sum_{i,\alpha,\beta} |\frac{\partial}{\partial A_i(\alpha\beta)} \frac{1}{N} \operatorname{tr} P(\mathbf{A})|^2)^{\frac{1}{2}}
$$

$$
= \sup_{\{\|\mathbf{A}\| \leqslant M\}} (\sum_{i,\alpha,\beta} |\frac{1}{N}(D_i P(\mathbf{A}))_{\alpha\beta}|^2)^{\frac{1}{2}}
$$

$$
= \sup_{\{\|\mathbf{A}\| \leqslant M\}} (\sum_i \frac{1}{N^2} \operatorname{tr}(D_i P (D_i P)^*))^{\frac{1}{2}}
$$

$$
\leqslant \frac{C}{\sqrt{N}} \sup_i \|D_i P\|_M \leqslant \frac{C'}{\sqrt{N}} \|P\|_{M'}
$$

where we choose $M' > M$ and use the continuity of the cyclic derivative. We want to use a concentration inequality, thus we define an extension φ of $\hat{\mu}^N(P)$ on $\mathcal{H}_N(\mathbb{C})^m$ which is equal to $\hat{\mu}^N(P)$ on $\{\|\mathbf{A}\| \leqslant M\}$ and with the same lipschitz constant. For example, one can define:

$$
\phi(\mathbf{A}) = \sup_{\|\mathbf{B}\| \leqslant M} (\frac{1}{N} \operatorname{tr}(P(\mathbf{B})) - \frac{C'}{\sqrt{N}} \|P\|_{M'} \|\mathbf{A} - \mathbf{B}\|_2).
$$

It is an easy exercise to check that ϕ is a $\frac{C'}{\sqrt{N}} \|P\|_{M'}$-lipschitz function which coincide with $\hat{\mu}^N(P)$ on $\{\|\mathbf{A}\| \leqslant M\}$. Now observe that with our hypothesis, for all $\mathbf{t}$ in $B_{\eta,c}$, μ_V^N satisfy a log-sobolev inequality with constant Nc. Thus, according to Herbst's argument (see [1]) we obtain a concentration's inequality, for all λ-lipschitz function f,

$$
\mu_V^N(|f - \mu_V^N(f)| > \varepsilon) \leqslant 2e^{-\frac{Nc\varepsilon^2}{\lambda^2}}.
$$

Therefore,

$$
\mu_V^N[|N(\hat{\mu}_-^N(P) - E[\hat{\mu}_-^N(P)]| > \varepsilon] \leqslant 2e^{-\frac{C\varepsilon^2}{\|P\|_{M'}^2}}
$$

and

$$
E[|N(\hat{\mu}_-^N(P) - E[\hat{\mu}_-^N(P)]|^\ell] \leqslant C' \|P\|_{M'}^\ell.
$$

Thus with (9)

$$
E[|N(\hat{\mu}^N(P) - E[\hat{\mu}^N(P)])|^\ell] \leq 2^\ell (C' + C) \|P\|_M^\ell
$$

and we conclude with (7) and (8).

$\square$

We now try to find some relation between the $N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)]$'s which generalize (6). Remember that, for ℓ odd, those quantities vanish when N goes to infinity. Thus in order to obtain non-trivial limits, we have to distinguish the normalisation according to the parity of ℓ.

For $\ell \geq 1$ we define:

$$\hat{\ell} = \begin{cases} \ell + 1 & \text{if } \ell \text{ is odd} \\ \ell & \text{otherwise.} \end{cases}$$

Note that $\hat{\ell}$ is always an even integer. We now define a function from $\sqcup_{\ell \in \mathbb{N}} \mathbb{C}\langle X_1, \cdots, X_m \rangle^{\otimes \ell}$ to $\mathbb{C}$,

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = E_{\mu_V^N}[N^{\hat{\ell}}(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)].$$

On $\mathbb{C}\langle X_1, \cdots, X_m \rangle^{\otimes \ell}$, this is a ℓ-linear symmetric function which is tracial in each P_r. Our convention will be that for λ in $\mathbb{C} = \mathbb{C}\langle X_1, \cdots, X_m \rangle^{\otimes 0}$, $\nu^N(\lambda) = \lambda$. The relation that will appear as our main tool are the aim of the next property. In a tensor product $P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell$ denotes the tensor product of $P_1, \ldots, P_{r-1}, P_{r+1}, \ldots, P_\ell$ i.e. the term P_r is omitted.

Property 3.2 *For all ℓ, for all polynomial $P_1, \ldots, P_\ell$, for all N, if ℓ is even*

$$\nu^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu_{\mathbf{t}}(D_i \overline{P} D_i P_r) \nu^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \frac{1}{N^2} \sum_{i,r} \nu^N(D_i \overline{P} D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \frac{1}{N^2} \sum_i \nu^N(\partial_i D_i \overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)$$

and if ℓ is odd

$$\nu^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu_{\mathbf{t}}(D_i \overline{P} D_i P_r) \nu^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{i,r} \nu^N(D_i \overline{P} D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \nu^N(\partial_i D_i \overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)$$

This property is the generalization of the equation (6). One may wonder why we stress so much the difference between the odd and the even case.

The point is to keep in mind which terms are of order 1 and which are negligible. In view of this, the ν_1^N are convenient as they should all be of order 1 and thus the previous equation will lead us to find their limit by induction.

Proof.

To sum up the property in a shorter way, we have to prove that for all ℓ for all polynomials $P_1,\dots,P_\ell$ and for all N,

$$
N^\ell E[(\hat{\mu}^N - \mu)^{\otimes\ell}(\Xi P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)]
$$
$$
= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P_1}D_iP_r)N^{\ell-2}E[(\hat{\mu}^N - \mu)^{\otimes\ell-2}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)
$$
$$
+ \sum_{i,r} N^{\ell-2}E[(\hat{\mu}^N - \mu)^{\otimes\ell-1}(D_i\overline{P_1}.D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)]
$$
$$
+ \sum_i N^\ell E[(\hat{\mu}^N - \mu)^{\otimes\ell+1}(\partial_i D_i\overline{P_1} \otimes P_2 \otimes \cdots \otimes P_\ell)].
$$

We will use the integration by part formula:

$$
\int x f(x)e^{-x^2/2}dx = \int f'(x)e^{-x^2/2}dx.
$$

We generalize this formula into

$$
\int \mathrm{tr}(A_iP)f(\mathrm{tr}\,Q)d\mu^N = \frac{1}{N}\sum_{\alpha,\beta}\int (\partial_{A_i(\alpha\beta)}P_{\beta\alpha})f(\mathrm{tr}\,Q)
$$
$$
+ P_{\beta\alpha}\partial_{A_i(\alpha\beta)}\,\mathrm{tr}\,Qf'(\mathrm{tr}\,Q)d\mu^N.
$$

Two useful computations show the importance of the non-commutative derivatives and their links with the derivation of polynomials of hermitian matrices: if P is a monomial,

$$
\sum_{\alpha\beta}\partial_{A_i(\alpha\beta)}P_{\beta\alpha} = \sum_{\alpha\beta}\sum_{P=RX_iS}R_{\beta\beta}S_{\alpha\alpha} = \mathrm{tr}\otimes\mathrm{tr}(\partial_i P)
$$

and

$$
\partial_{A_i(\alpha\beta)}\,\mathrm{tr}\,P = \sum_{P=RX_iS,\gamma}R_{\gamma\beta}S_{\alpha\gamma} = (DP)_{\alpha\beta}.
$$

Thus, for $P_1, \ldots, P_\ell$ polynomials:

$$N^\ell E[\hat{\mu}^N (X_i P_1)(\hat{\mu}^N - \mu)^{\otimes \ell - 1}(P_2 \otimes \cdots \otimes P_\ell)]$$

$$= N^\ell E[(\hat{\mu}^N \otimes \hat{\mu}^N)(\partial_i P_1)(\hat{\mu}^N - \mu)^{\otimes \ell - 1}(P_2 \otimes \cdots \otimes P_\ell)]$$

$$+ \sum_r N^{\ell - 2} E[\hat{\mu}^N (P_1 D_i P_r)(\hat{\mu}^N - \mu)^{\otimes \ell - 1}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)]$$

$$- N^\ell E[\hat{\mu}^N (P_1 D_i V)(\hat{\mu}^N - \mu)^{\otimes \ell - 1}(P_2 \otimes \cdots \otimes P_\ell)].$$

Now remember that according to Schwynger Dyson's equation we have:

$$\mu((X_i + D_i V)P_1) - \mu \otimes \mu(\partial_i P_1) = 0$$

Then, we substract the two equalities and use the identity

$$\hat{\mu}^N \otimes \hat{\mu}^N - \mu \otimes \mu = (\hat{\mu}^N - \mu)(I \otimes \mu + \mu \otimes I) + (\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu)$$

to obtain

$$N^\ell E[(\hat{\mu}^N - \mu)((X_i + D_i V)P_1 - (I \otimes \mu + \mu \otimes I)\partial_i P_1)$$

$$(\hat{\mu}^N - \mu)^{\otimes \ell - 1}(P_2 \otimes \cdots \otimes P_\ell)]$$

$$= N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell + 1}(\partial_i P \otimes P_2 \otimes \cdots \otimes P_\ell)]$$

$$+ \sum_r N^{\ell - 2} E[(\hat{\mu}^N - \mu)^{\otimes \ell - 1}(P D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)]$$

$$+ \sum_r N^{\ell - 2} \mu(P D_i P_r) E[(\hat{\mu}^N - \mu)^{\otimes \ell - 2} \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)].$$

To get the result it is now sufficient to aplly this equality with $P_1 = D_i \bar{P}$ and then to sum on i.

$$\square$$

This property gives us some precious hints on the limit ν of the ν^N. It should satisfy the "limit equation", if ℓ is even

$$\nu(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu_{\mathbf{t}}(D_i \overline{P} D_i P_r)\nu(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (10)$$

and if ℓ is odd

$$\nu(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu_{\mathbf{t}}(D_i \overline{P} D_i P_r)\nu(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (11)$$

$$+ \sum_{i,r} \nu(D_i \overline{P} D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \nu(\partial_i D_i \overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell).$$

Hopefully, we will be able to study the solutions ν of these equations. In fact, following [7] we would be able to prove that for $R, L > 0$, there exists $\varepsilon > 0$ such that for $|\mathbf{t}| < \varepsilon$ there exists an unique $\nu : \sqcup_{\ell=0}^{2L} \mathbb{C}\langle X_1, \cdots, X_m \rangle^{\otimes \ell} \to \mathbb{C}$ linear on each set $\mathbb{C}\langle X_1, \cdots, X_m \rangle^{\otimes \ell}$ with support bounded by R and which satisfy each of the previous equation for $\ell \leq 2L$. But we will proceed in a different way. Looking at these equations, we will try to recognize in them some relations between enumeration of combinatorial objects. This is the aim of the next section.

4   Maps of high genus

In this section, we describe the combinatorials objects that appear in the asymptotic of our measure. Remember that it was shown in [7] that the first asymptotic can be viewed as a generating function for the enumeration of planar maps with vertices of a given type.

First, we choose m colors $\{1, \cdots, m\}$, one for each variable X_i. A star must be thought as the neighbourhood of a vertex in a plane graph. More precisely, it is a vertex with the half-edges coming out of it. One of these half-edge is distinguished and starting from it the other one are clockwisely ordered. Besides, each of these half-edges is colored.

We say that a star is of type q for a monomial $q = X_{i_1} \cdots X_{i_p}$ if it has p half-edges, the first half-edge is distinguished and of color i_1 and then in the clockwise order the second half-edge is of color i_2, the third of color i_3, ..., the p-th of color i_p. This gives a bijection between monomials and stars.

The combinatorial objects that will appear in the asymptotic of our matrix model are maps. A map is a connected graph on a compact orientated connected surface such that edges do not cross each other and faces are homeomorphic to discs. We will consider edge-colored maps such that each vertex as a distinguished edge going out of it so that we can associate a star and a well defined type to any vertex. The genus of the map is the genus of the surface. We will count maps up to homeomorphism of the surface which preserves the graph.

The typical way to construct a map is to put some stars $q_1, \cdots, q_p$ on a surface of genus g. Then we consider all the half-edges that goes outside the stars and glue them two by two while respecting the following constraints:

- Two half-edges can only be glued if they are of the same color

- The edges created by gluing two half-edges mustn't cross any other edge.

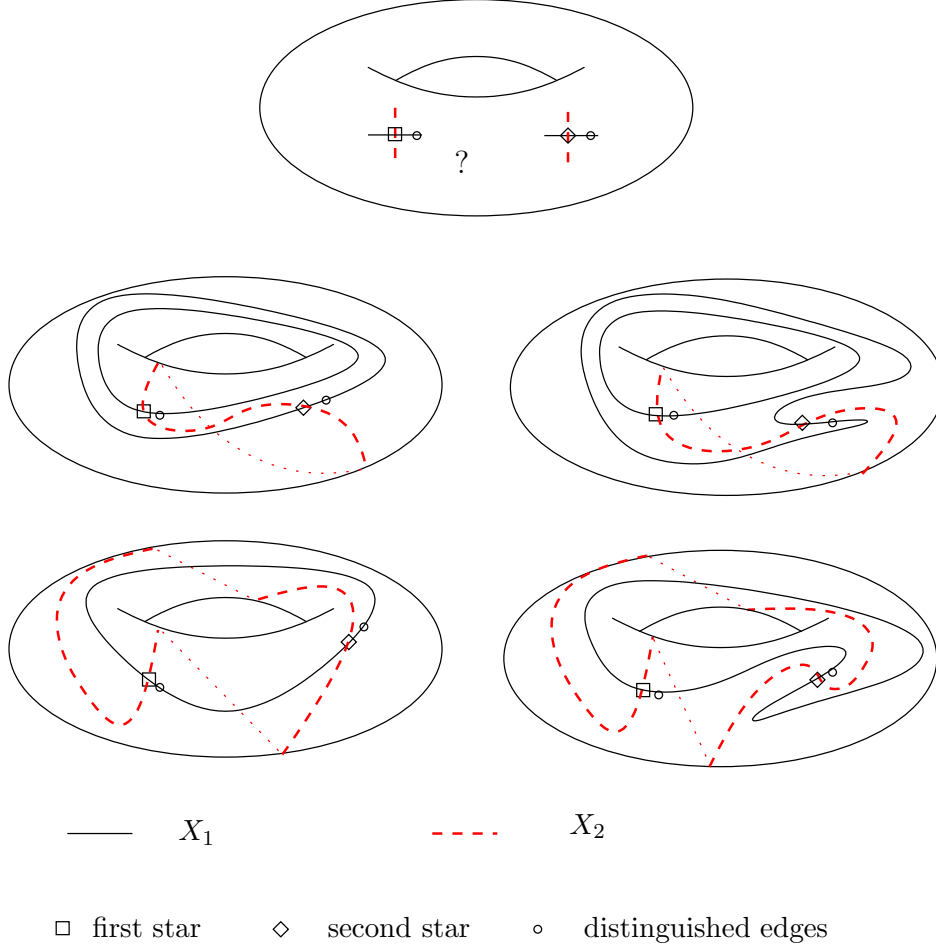• At the end of the process faces must be homeomorphic to discs.

| —— | X_1 | - - - - | X_2 |

□ first star   ◇ second star   ∘ distinguished edges

Figure 1: Maps of genus 1 above two stars of type $X_1 X_2 X_1 X_2$.

For example, one can ask how many maps of genus 1, we can construct above two stars of type $X_1 X_2 X_1 X_2$. The answer as shown in figure 1 is 4. Note that as faces are homeomorphic to discs, it is sufficient to know which pairs of half-edges are glued together to build the map.

Definition 4.1 *For ℓ in $\mathbb{N}$, P a monomial and integers $\mathbf{k} = (k_1, \cdots, k_n)$, let $\mathcal{M}^{\mathbf{k}}(P)$ be the number of maps of genus 0 (or planar maps) with for all i, k_i stars of type q_i and one of type P where $q_1, \cdots, q_n$ are the monomials which appear in the potential V.*

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We could define the same kind of quantities with the condition of being of genus g for $g > 1$ but then we won't be able to find any closed relation of induction between these quantities. In order to get relation induction on enumeration of maps we follow an idea of Tutte (see [9]). We try to decompose a map in smaller ones by contracting one edge (Note that Tutte used to work on the dual of the graph we are considering, thus his operation is a little different).

Imagine a map of genus 1 with a root of type $P = XRXS$ and that the two half-edges corresponding to the X are glued together. Imagine also that the loop resulting from this operation is not retractable on the surface. How does the contraction of this edge decomposes the map? Now R and S are separated by that loop, we will have to remind these two monomials. That's why we will introduce maps above a root of type $R \otimes S$. Besides R and S must be linked together, otherwise there would be a face (touched by the loop) which is not a disc, something to avoid for a map.

Thus we define some more complex vertices which will appear when we will try to decompose our maps. Let $P_1, \ldots, P_\ell$ be a family of monomials. We associate to this family a bunch of $\ell - 1$ circles such that outside the circles we put the half-edges of P_1 and in the m-th circle we put the half-edges of a star of type P_{m+1} going out of the central point and in the same order. This object will be called the root and we will name each P_r a vertex of the root (look at figure 2, to see a root of type $X_1 X_2 X_1 X_2 \otimes X_1^2 \otimes X_2 X_1$).
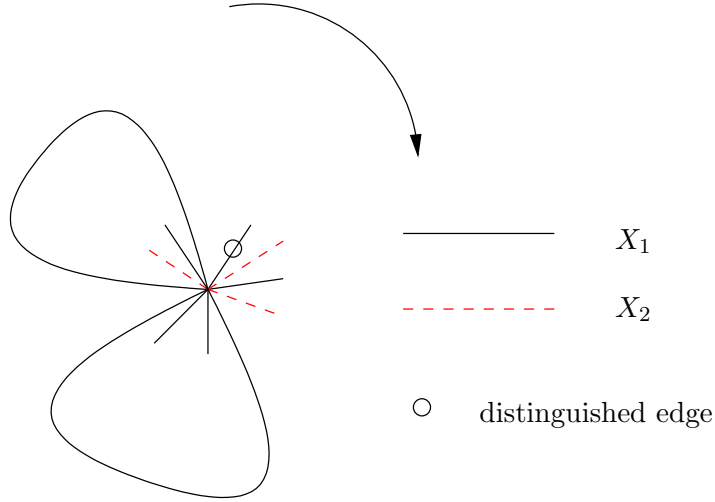

Figure 2: Root of type $X_1 X_2 X_1 X_2 \otimes X_1^2 \otimes X_2 X_1$.

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This corresponds to a star coming from a vertex which have ℓ prescribed loops, the m-th having the germs of edges corresponding to a star of type P_r. Now we construct maps with a root of type $P_1 \otimes \cdots \otimes P_\ell$ and some other vertices of type q_i. We say that such a map is minimal if when we cut the surface along the $\ell - 1$ loops of the root, we do not obtain any component homeomorphic to a disc. This means that for any P_i the component of P_i is not planar i.e. either it is linked to another P_j or it is linked to some other vertices in a way that can't be embedded on a sphere.

Definition 4.2 *For ℓ, g in $\mathbb{N}$, a family of monomials $P_1, \cdots, P_\ell$ and integers $\mathbf{k} = (k_1, \cdots, k_n)$, let $\mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell)$ be the number of minimal maps of genus g with a root of type $P_1 \otimes \cdots \otimes P_\ell$ and for all i, k_i stars of type q_i.*

For example for $V = tX_1X_2X_1X_2$, the figure 1 shows that $\mathcal{M}_1^1(X_1X_2X_1X_2)$ the number of minimal maps of genus 1 with a star of type $X_1X_2X_1X_2$ and a root of type $X_1X_2X_1X_2$ is 4.

We extend by linearity $\mathcal{M}_{\mathbf{k}}$ and $\mathcal{M}_g^{\mathbf{k}}$ so that we can compute them on polynomials P_i instead of monomials and we define the power series for these enumerations:

$$\mathcal{I}(P) = \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}}(P)$$

and

$$\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell).$$

By convention we define for λ in $\mathbb{C}$, $\mathcal{I}_g(\lambda) = \lambda \mathbb{1}_{g=0}$ and $\mathcal{I}_g \equiv 0$ ig $g < 0$.
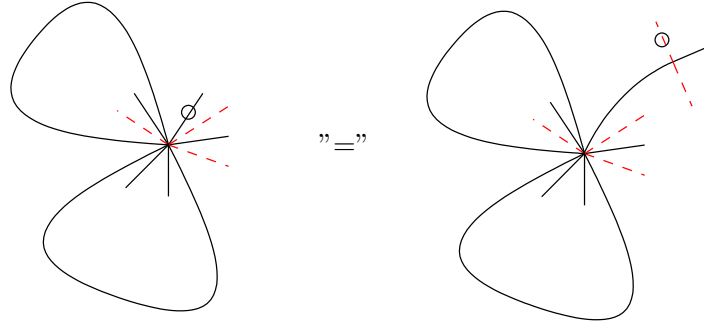
Recall that it was proved in [7] that for $\mathbf{t}$ sufficiently small $\mathcal{I}(P) = \mu(P)$ for all P. This was proved using the fact that these two quantities satisfy the same induction relation. The induction relation for the enumeration of maps where given by a decomposition of maps following the strategy of Tutte. We now try to generalize this fact and we begin by looking at the relation given by decomposing maps. First, some values can be directly computed

$$\mathcal{M}_g^{\mathbf{k}}(1 \otimes P_2 \otimes \cdots \otimes P_\ell) = \mathbb{1}_{g=\mathbf{k}=\ell=0}$$

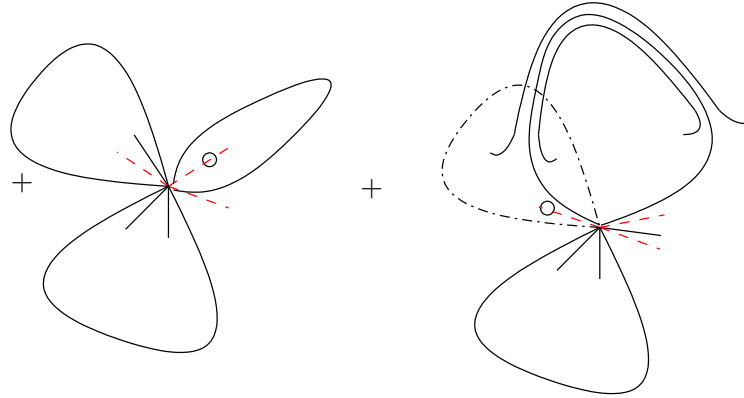because the component of 1 is automatically planar.

We now want to count maps that contribute to $\mathcal{M}_g^{\mathbf{k}}(X_iP_1 \otimes \cdots \otimes P_\ell)$ with P_i monomials. We look at the first half-edge of the root $X_iP_1 \otimes \cdots \otimes P_\ell$ and see where it is glued. Remember that it must not be planar.

Then three cases may occur (see figure 3):

Where can we glue the first half-edge ?

(1): To an other vertex.

(2): To a half-edge of the same vertex.

(3): To another vertex of the root.

————	X_1
- - - - - -	X_2

○   distinguished edge

Figure 3: The decomposition process for maps.

1. Either (upper right picture in fig 3) the half-edge is glued to a vertex of type $q_j = RX_iS$ for a given j. First we have to choose between the k_j vertices of this type, then we contract the edge coming from this

gluing to form a vertex of type SRP_1. This creates

$$\sum_{1 \leq j \leq n, k_j \neq 0} k_j \mathcal{M}_g^{k_1, \cdots, k_j - 1, \cdots, k_n} (D_i q_j P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

possibilities.

2. The second case (bottom left picture in fig 3) occurs if the half-edge is glued to another half-edge of $P_1 = RX_iS$. It cuts P_1 in two: R and S. It occurs for all decomposition of P_1 into $P_1 = RX_iS$. To write the expression that will arise in a more convenient way, we will use the non-commutative derivative ∂ which satisfy for P a monomial

$$\partial_i P = \sum_{P = RX_iS} R \otimes S.$$

We are now left with two separate circles, one for R and one for S. Either both are non-planar which leads to

$$\sum_{P_1 = RX_iS} \mathcal{M}_g^{\mathbf{k}}(R \otimes S \otimes P_2 \otimes \cdots \otimes P_\ell) = \mathcal{M}_g^{\mathbf{k}}(\partial_i P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

possibilities or one of the component is planar then the two components can not be linked thus we have to share the vertices of type q_i between them. They are $\binom{\mathbf{k}}{\mathbf{k'}} = \prod_j \binom{k_j}{k'_j}$ ways of choosing for all j, k'_j vertices of type q_j for the component of R.

If the component of R is planar and the one of S is not this leads to

$$\sum_{\mathbf{k'} + \mathbf{k''} = \mathbf{k}} \binom{\mathbf{k}}{\mathbf{k'}} \mathcal{M}_g^{\mathbf{k''}}((\mathcal{M}^{\mathbf{k'}} \otimes I)(\partial_i P_1) \otimes \cdots \otimes P_\ell)$$

possibilities or S is planar and R is not,

$$\sum_{\mathbf{k'} + \mathbf{k''} = \mathbf{k}} \binom{\mathbf{k}}{\mathbf{k'}} \mathcal{M}_g^{\mathbf{k'}}((I \otimes \mathcal{M}^{\mathbf{k''}})(\partial_i P_1) \otimes \cdots \otimes P_\ell)$$

possibilities.

3. The last case occurs if the half-edge is glued with another vertex $P_r = RX_iS$ of the root. This create a vertex of type $D_i P_r P_1$. Note that the edge can not cross the circles of the root so it must go through a handle of the surface thus it changes the genus by one. But this

vertex is now free from the condition of non planarity so it can either be planar and thus be separate from the other vertices of the root:

$$\sum_{2 \leq m \leq l, \mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}^{\mathbf{k}'}(D_i P_r P_1) \mathcal{M}^{\mathbf{k}''}_{g-1}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

possibilities or it may still be non-planar:

$$\sum_{2 \leq m \leq \ell} \mathcal{M}^{\mathbf{k}}_{g-1}(D_i P_r P_1 \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

possibilities.

We can sum these identities and then sum on the k_i's to obtain the following equality, for all g, for all $P_1 \ldots, P_\ell$,

$$\mathcal{I}_g(X_i P_1 \otimes \cdots \otimes P_\ell) = \sum_j (-t_j) \mathcal{I}_g(D_i q_j P_1 \otimes \cdots \otimes P_\ell)$$

$$+ \mathcal{I}_g((I \otimes I + \mathcal{I} \otimes I + I \otimes \mathcal{I}) \partial_i P_1 \otimes \cdots \otimes P_\ell)$$

$$+ \sum_{m \geq 2} \mathcal{I}_{g-1}((I + \mathcal{I}) D_i P_r P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

We can reformulate this by applying it to $P_1 = D_i \bar{P}$ and then summing on i:

$$\mathcal{I}_g(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{g-1}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \quad (12)$$

$$+ \sum_{m \geq 2, i} \mathcal{I}_{g-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \mathcal{I}_g(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell).$$

where we used the identity $\mathcal{I} = \mu$. Note that maps that appear in the enumeration must satisfy the condition of non-planarity, this imposes a high genus. We have to break the "planarity" of ℓ components, this can't be done without at least $[\frac{\ell+1}{2}]$ handles on the surface (each handle allow one edge to cross from one vertex of the root to another one, breaking the planarity of two components at most). Thus if $g < Ent(\frac{\ell+1}{2})$, $\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = 0$. This allow us to write the previous equation in a special case which will appear to be useful, if ℓ is even,

$$\mathcal{I}_{\frac{\ell}{2}}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{\frac{\ell-2}{2}}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell). \quad (13)$$

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Thus for ℓ even, $\mathcal{I}_{\frac{\ell}{2}}$ satisfy the limit equation (10) of the matrix model. One can easily check that for ℓ odd $\mathcal{I}_{\frac{\hat{\ell}}{2}} = \mathcal{I}_{\frac{\ell+1}{2}}$ satisfy also the limit equation (11):

$$\mathcal{I}_{\frac{\ell+1}{2}}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m \geq 2, i} \mathcal{I}(D_i P_r D_i \overline{P}) \mathcal{I}_{\frac{(\ell-2)+1}{2}}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{m \geq 2, i} \mathcal{I}_{\frac{\ell-1}{2}}(D_i P_r D_i \overline{P} \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \mathcal{I}_{\frac{\ell+1}{2}}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell).$$

We can deduce from these identities a control on these enumerations:

Lemma 4.3 *For all $g \geq 0$, there exists $\varepsilon > 0$ such that for $\mathbf{t} \in B(0, \varepsilon)$, $\mathcal{I}^g$ is absolutely convergent and has a bounded support i.e. there exists $M > 0$ such that for all polynomials $P_1, \cdots, P_\ell$,*

$$|\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell)| \leqslant \|P_1\|_M \cdots \|P_\ell\|_M$$

.

Proof.

It is sufficient to show that for all $g \geq 0$, there exists $A_g, B_g > 0$ such that for all h, for all monomials $P_1, \ldots, P_\ell$, and all integers k_i:

$$\frac{\mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell)}{\prod_i k_i!} \leq A_g^{\Sigma \deg P_i} B_g^{\Sigma k_i}$$

This is easy by induction using the decomposition of maps.

$\square$

Finally we need to know the effect of derivation on these generating function. In fact, derivation adds some vertices to the enumeration.

Lemma 4.4 *For all $\mathbf{j} = (j_1, \cdots, j_n)$,*

$$\mathcal{D}_{\mathbf{j}} \mathcal{I}(P) = (-1)^{\mathbf{j}} \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}+\mathbf{j}}(P)$$

and

$$\mathcal{D}_{\mathbf{j}} \mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = (-1)^{\mathbf{j}} \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}+\mathbf{j}}(P_1 \otimes \cdots \otimes P_\ell).$$

Besides, these series are absolutely convergent and has a bounded support.

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Proof.

The proof is straightforward, $\mathcal{I}, \mathcal{I}_g$ are analytic in a neightbourhood of the origin thus their derivatives are analytic and their series are given by differentiating term by term $\mathcal{I}$ and $\mathcal{I}_g$.

$\square$

Thus derivatives fix some vertices in the enumeration a fact often used in combinatorics to find relation between generating functions of graphs.

5   High order observable

We have already seen that $\nu(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\hat{\ell}}{2}}(P_1 \otimes \cdots \otimes P_\ell)$ satisfy the limit equation of ν^N. This is our candidate for the limit of the ν^N's. In fact this suggests a statement closely related to 1.1.

Property 5.1 *For all ℓ, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all polynomials $P_1, \ldots, P_\ell$,*

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\hat{\ell}}{2}}(P_1 \otimes \cdots \otimes P_\ell) + \frac{1}{N^2}\mathcal{I}_{\frac{\hat{\ell}}{2}+1}(P_1 \otimes \cdots \otimes P_\ell) +$$
$$\cdots + \frac{1}{N^{2g}}\mathcal{I}_{\frac{\hat{\ell}}{2}+g}(P_1 \otimes \cdots \otimes P_\ell) + o(\frac{1}{N^{2g}}).$$

To prove this we have to define all the correction to the convergence. We define $\nu_1^N = \nu^N$ and by induction on h, for all N, all polynomials $P_1, \ldots, P_\ell$,

$$\nu_{h+1}^N(P_1 \otimes \cdots \otimes P_\ell) = N^2(\nu_h^N - \mathcal{I}_{\frac{\hat{\ell}}{2}+h-1})(P_1 \otimes \cdots \otimes P_\ell).$$

Those quantities satisfy also some induction relation similar to those of property 3.2

Property 5.2 *For all $h \geqslant 2$, ℓ in $\mathbb{N}$, for all polynomial $P_1, \ldots, P_\ell$, for all N, the "finite Schwynger-Dyson's equation of order h" $(\mathbf{SD}_{h,\ell}^N)$ is satisfied by ν^N: if ℓ is even*

$$\nu_h^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu(D_i\overline{P}D_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{i,r} \nu_{h-1}^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (\mathbf{SD}_{h,\ell}^N)$$

$$+ \sum_i \nu_{h-1}^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)$$

and if ℓ is odd

$$\nu_h^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu(D_i\overline{P}D_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{i,r} \nu_h^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (\mathbf{SD}_{h,\ell}^N)$$

$$+ \sum_i \nu_h^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)$$

Proof.

Remember that we have shown in Property 3.2, for ℓ even

$$\nu_1^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu_1^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \frac{1}{N^2} \sum_{i,r} \nu_1^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \frac{1}{N^2} \sum_i \nu_1^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)$$

and according to (13)

$$\mathcal{I}_{\frac{\ell}{2}}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m\geq 2,i} \mathcal{I}(D_iP_rD_i\bar{P})\mathcal{I}_{\frac{\ell-2}{2}}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

Thus if we substract these two equalities and multiply the result by N^2 we obtain $(\mathbf{SD}_{2,\ell}^N)$ (Observe that with our convention $\nu^N(\lambda) = \mathcal{I}_0(\lambda)$).

$$\nu_2^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu(D_i\overline{P}D_iP_r)\nu_2^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{i,r} \nu_1^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \nu_1^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell).$$

Now suppose that for ℓ even, $h \geqslant 2$, for all polynomial $P_1,\ldots,P_\ell$, for all N, $(\mathbf{SD}_{h,\ell}^N)$ is satisfied. Then according to (12):

$$\mathcal{I}_{\frac{\ell}{2}+h-1}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_i \mathcal{I}_{\frac{(\ell+1)+1}{2}+h-2}(\partial_i D_i\bar{P} \otimes \cdots \otimes P_\ell)$$

$$+ \sum_{r\geq 2,i} \mathcal{I}_{\frac{(\ell-1)+1}{2}+h-2}(D_iP_rD_i\bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{r\geq 2,i} \mu(D_iP_rD_i\bar{P})\mathcal{I}_{\frac{(\ell-2)}{2}+h-1}(D_iP_rD_i\bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

and this can be translated into

$$\mathcal{I}_{\frac{\hat{\ell}}{2}+h-1}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_i \mathcal{I}_{\widehat{\frac{\ell+1}{2}}+h-2}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell)$$

$$+ \sum_{r\geq 2,i} \mathcal{I}_{\widehat{\frac{\ell-1}{2}}+h-2}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{r\geq 2,i} \mu(D_i P_r D_i \bar{P})\mathcal{I}_{\widehat{\frac{\ell-2}{2}}+h-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

Substracting this equality from $(\mathbf{SD}_{h,\ell}^N)$ we get $(\mathbf{SD}_{h+1,\ell}^N)$. This proves by induction $(\mathbf{SD}_{h,\ell}^N)$ for all h, and for all ℓ even.

We proceed in the same way for ℓ odd. Observe that the equation for ℓ odd and $h = 1$ is satisfied according to Property 3.2. Then observe that for ℓ odd, $(\mathbf{SD}_{h+1,\ell}^N)$ can be obtained by substracting (12) with $g = \frac{\ell+1}{2} + h - 1$

$$\mathcal{I}_{\frac{\hat{\ell}}{2}+h-1}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_i \mathcal{I}_{\widehat{\frac{\ell+1}{2}}+h-2}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell)$$

$$+ \sum_{r\geq 2,i} \mathcal{I}_{\widehat{\frac{\ell-1}{2}}+h-2}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{r\geq 2,i} \mu(D_i P_r D_i \bar{P})\mathcal{I}_{\widehat{\frac{\ell-2}{2}}+h-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

from $(\mathbf{SD}_{h,\ell}^N)$.

$\square$

6   Asymptotic of the matrix model

The issue with the previous relations is that they only give us the moments of products of polynomials such that the first polynomial is in the image of Ξ. Thus we need to invert Ξ. We define the operator norm with respect to $\|.\|_M$:

$$|||A|||_M = \sup_{\|P\|_M \leqslant 1} \|AP\|_M.$$

In [7], we give some estimates on the operator norm of Ξ.

Lemma 6.1   *1. The operator Ξ_0 is invertible on $\mathbb{C}\langle X_1, \cdots, X_m \rangle$.*

2. *There exists $M_0 > 0$ such that for all $M > M_0$, the operators Ξ_2, Ξ_0 and Ξ_0^{-1} are continuous and their norm are uniformly bounded for $\mathbf{t}$ in B_η.*

3. *For all polynomials P, $\deg \Xi_0^{-1} P \leqslant \deg P$ and $\deg \Xi_1 P \leqslant \deg P + \deg V - 2$*

4. *For all $\varepsilon, M > 0$, there exists $\eta_\varepsilon > 0$ such for $|\mathbf{t}| < \eta_\varepsilon$, Ξ_1 is continuous on $\mathbb{C}\langle X_1, \cdots, X_m \rangle$ and $|||\Xi_1|||_M \leqslant \varepsilon$.*

The last step to proves Theorem 1.1 is to control the ν_N^h.

Lemma 6.2 *For all $\ell, h \in \mathbb{N}^*$, $\alpha > 0$, there exists $C, \eta, M_0 > 0$, such that for all $\mathbf{t} \in B_{\eta,c}$, $M > M_0$ and all polynomials $P_1, \ldots, P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$,*

$$|\nu_h^N(P_1 \otimes \cdots \otimes P_\ell)| \leq C\|P_1\|_M \cdots \|P_\ell\|_M$$

Proof.

The case $h = 1$, ℓ even is a direct consequence of Lemma 3.1. We treat the other cases by induction using Property 5.2. As the equations are different according to the parity of ℓ, we have to be careful: we prove the result by an induction on h and for a fixed h we deal first with the case ℓ even and then with the case ℓ odd (Note that both time we will do an induction on ℓ). Now we choose $\ell, h \in \mathbb{N}^*$, $\alpha > 0$ and polynomials $P_1, \ldots, P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$.

Then, the idea to nearly invert Ξ on a polynomial P is to approximate P_1 by $\Xi Q_n = (\Xi_0 + \Xi_1) Q_n$ with

$$Q_n = \sum_{k=0}^{n-1} (-\Xi_0^{-1}\Xi_1)^k \Xi_0^{-1} P_1.$$

The remainder is

$$R_n = P_1 - \Xi Q_n = (-\Xi_1\Xi_0^{-1})^n P_1.$$

As Ξ_1 is the multiplication by a derivative of V it should have a small norm and the remainder should be easily controlled.

We can make the decomposition:

$$\nu_h^N(P_1 \otimes \cdots \otimes P_\ell) = \nu_h^N(\Xi Q_n \otimes \cdots \otimes P_\ell) + \nu_h^N(R_n \otimes \cdots \otimes P_\ell) \qquad (14)$$

Now we let n goes to infinity with N, for example $n = [\sqrt{N}]$. It is important that n goes to infinity no too slowly but we must have $n = O(\sqrt{N})$ in order to use all the induction hypothesis. An important fact is that the degrees of R_n and Q_n are $O(\sqrt{N})$ since $\Xi_0^{-1}\Xi_1$ change the degree by at most $D - 2$.

We first control the term with R_n, by definition of the ν_N^h,

$$\nu_h^N(R_n \otimes \cdots \otimes P_\ell) = (N^{2(h-1)}\nu_1^N + N^{2(h-1)}\mathcal{I}_{\frac{\hat{\ell}}{2}} + \cdots + \mathcal{I}_{\frac{\hat{\ell}}{2}+h-1})(R_n \otimes \cdots \otimes P_\ell).$$

Each of the $\mathcal{I}_g$ are compactly supported according to Lemma 4.3 so that if η is sufficiently small, for $\mathbf{t}$ in $B_{\eta,c}$, $\mathcal{I}_{\frac{\hat{\ell}}{2}}, \ldots, \mathcal{I}_{\frac{\hat{\ell}}{2}+h-1}$ are convergent and we can take M bigger than the radius of their support. Besides, Property 3.1 shows that for polynomials P of degree of order $N^{\frac{1}{2}}$,

$$|\nu_1^N(P_1 \otimes \cdots \otimes P_\ell)| \leqslant CN\|P_1\|_M \cdots \|P_\ell\|_M.$$

Thus according to Lemma 6.1, for η small, $\|R_n\|_M \leqslant |||\Xi_1\Xi_0^{-1}|||^n\|P_1\|_M$ decrease exponentially fast in n and this uniformly for $\mathbf{t} \in B_{\eta,c}$. Then since $n \sim \sqrt{N}$

$$|\mathcal{I}_g(R_n \otimes \cdots \otimes P_\ell)| \leqslant \|R_n\|_M\|P_2\|_M \ldots \|P_\ell\|_M \leqslant Ce^{-C'\sqrt{N}}\|P_1\|_M \ldots \|P_\ell\|_M.$$

Thus,

$$|\nu_h^N(R_n \otimes \cdots \otimes P_\ell)| \leqslant N^{2h}Ce^{-C'\sqrt{N}}\|P_1\|_M \ldots \|P_\ell\|_M \tag{15}$$

and $N^{2h}Ce^{-C'\sqrt{N}}$ is bounded.

Finally we have to deal with $\nu_h^N(\Xi Q_n \otimes \cdots \otimes P_\ell)$. We can use $(\mathbf{SD}_{h,\ell}^N)$:

$$\nu_h^N(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,m} \mu_{\mathbf{t}}(D_i\bar{Q}_nD_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$\tag{16}$$

$$+ \sum_{i,m} \nu_{h-\mathbb{1}_{\ell\text{ even}}}^N(D_i\bar{Q}_nD_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \nu_{h-\mathbb{1}_{\ell\text{ even}}}^N(\partial_iD_i\bar{Q}_n \otimes P_2 \otimes \cdots \otimes P_\ell)$$

We now use the induction hypothesis. Indeed if h is even, on the right hand side either h decreases or h remains constant and ℓ decreases and remains even. If h is odd, either ℓ becomes even or ℓ decreases. Now, let C be an uniform bound on the norm of Ξ_0^{-1} (which exists according to Lemma 6.1) by definition of Q_n,

$$\|Q_n\|_M \leqslant \sum_{k=0}^{n-1} |||\Xi_0^{-1}\Xi_1|||_M^k |||\Xi_0^{-1}|||_M\|P\|_M \leqslant \frac{C}{1-C|||\Xi_1|||_M}\|P\|_M.$$

Thus, using Lemma 6.1, if η is sufficiently small, for $\mathbf{t}$ in $B_{\eta,c}$, $\|Q_n\|_M \leqslant 2\|P\|_M$. Note that

$$\deg Q_n \leqslant \deg P_1 + 2\sqrt{N}(D-1) \leqslant (\alpha + 2(D-1))\sqrt{N}.$$

We can now apply the induction hypothesis with $\alpha' = \alpha + 2(D-1)$, there exists M, C, η such that for $\mathbf{t}$ in $B_{\eta,c}$,

$$|\mu_{\mathbf{t}}(D_i\bar{Q}_nD_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)| \tag{17}$$
$$\leqslant C\|D_i\bar{Q}_nD_iP_r\|_M\|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|$$

where we have assumed that M is bigger than the radius of the support of $\mu_{\mathbf{t}}$ which is bounded according to (3). Besides, by the induction hypothesis we can obtain with the same constant,

$$|\nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(D_i\bar{Q}_nD_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)| \tag{18}$$
$$\leqslant C\|D_i\bar{Q}_nD_iP_r\|_M\|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|_M$$

and

$$|\nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(\partial_iD_i\bar{Q}_n \otimes P_2 \otimes \cdots \otimes P_\ell)| \tag{19}$$
$$\leqslant \|\partial_iD_i\bar{Q}_n\|_M\|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|_M.$$

Now remember, that if $M' > M$,

$$D_i : (\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_{M'}) \to (\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_M)$$

is continuous, thus

$$\|D_i\bar{Q}_nD_iP_r\|_M \leqslant \|D_i\bar{Q}_n\|_M\|D_iP_r\|_M$$
$$\leqslant C\|Q_n\|_{M'}\|P_r\|_{M'} \leqslant C\|P_1\|_{M'}\|P_r\|_{M'}$$

and

$$\|\partial_iD_i\bar{Q}_n\|_M \leqslant C\|P_1\|_{M'}.$$

If we use inequalities (17), (18) and (19) in the decomposition (16), we get

$$|\nu_h^N(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell)| \leqslant C\|P_1\|_{M'}\|P_2\|_{M'} \cdots \|P_\ell\|_{M'}.$$

Finally, we conclude with (15) and (14)

$$\square$$

Now, for all ℓ, h there exists $\eta > 0$ such that for $\mathbf{t} \in B_{\eta,c}$, for all polynomials $P_1, \cdots, P_\ell$,

$$\nu_g^N(P_1 \otimes \cdots \otimes P_\ell) = N^2(\nu_g^N - \mathcal{I}_{\frac{\hat{\ell}}{2}+g-1})(P_1 \otimes \cdots \otimes P_\ell)$$

is a bounded sequence for all $g \leqslant h+1$. Thus for all $g \leqslant h$, $\nu_g^N(P_1 \otimes \cdots \otimes P_\ell)$ goes to $\mathcal{I}_{\frac{\hat{\ell}}{2}+g-1}$ and

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\hat{\ell}}{2}} + \frac{1}{N^2}\mathcal{I}_{\frac{\hat{\ell}}{2}+1} + \cdots + + \frac{1}{N^{2h}}\mathcal{I}_{\frac{\hat{\ell}}{2}+h} + o(\frac{1}{N^{2h}}).$$

Thus Property 5.1 is proved. The special case $\ell = 1$ is exactly Theorem 1.2:

$$E[\hat{\mu}^N(P)] = \mathcal{I}(P) + \frac{1}{N^2}\nu_1^N(P)$$

$$= \mathcal{I}(P) + \frac{1}{N^2}\mathcal{I}_1(P) + \cdots + \frac{1}{N^{2g}}\mathcal{I}_g(P) + o(\frac{1}{N^{2g}}).$$

Thus we can prove Theorem 1.1.

Theorem 6.3 *For all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,*

$$F_{V_\mathbf{t}}^N = F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

and F^g is the generating function for maps of genus g associated with V:

$$F^g(\mathbf{t}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}^\mathbf{k}$$

where if $\mathbf{k} = (k_1, \cdots, k_n)$, $\mathcal{C}^\mathbf{k}$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

Proof.
Note that the estimate we get in Lemma 7.2 are uniform in $\mathbf{t}$ provided we are in $B_{\eta,c}$. Now observe that if $V_\mathbf{t}$ is c-convex then for α in $[0,1]$, $V_{\alpha\mathbf{t}}$ is c-convex if $c \leqslant 1$ and 1-convex if $c > 1$. Thus if $\mathbf{t}$ is in $B_{\eta,c}$, for all $0 < \alpha < 1$, $\alpha\mathbf{t} \in B_{\eta,\min(c,1)}$. This allow us to use Property 7.3 with an uniformly bounded remainder.

$$F_{V_\mathbf{t}}^N = \int_0^1 -E_{\mu_{V_{\alpha\mathbf{t}}}^N}[\hat{\mu}^N(V_\mathbf{t})]d\alpha$$

$$= -\int_0^1 \mu_{\alpha\mathbf{t}}(V_\mathbf{t})d\alpha - \frac{1}{N^2}\int_0^1 \nu_{\alpha\mathbf{t}}^N(\hat{\mu}^N(V_\mathbf{t}))d\alpha$$

$$= F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

with

$$F^0 = -\int_0^1 \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}}(V_{\mathbf{t}}) d\alpha$$

$$= \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{C}^{\mathbf{k}}$$

since it can be easily checked that

$$\sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-ut_i)^{k_i}}{k_i!} \mathcal{C}^{\mathbf{k}}$$

has the same derivative in u than $-\int_0^u \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}}(V_{\mathbf{t}}) d\alpha$.
With the same technique,

$$F^g = -\int_0^1 \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}}(V_{\mathbf{t}}) d\alpha$$

$$= \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{C}_g^{\mathbf{k}}$$

This proves the Theorem:

$$F_{V_{\mathbf{t}}}^N = F^0 + \cdots + \frac{1}{N^2 g} F^g + o(\frac{1}{N^{2g}}).$$

$\square$

7 Higher derivatives.

In this section we will show that one can derivate these expansions term by terms. Indeed, the family of the ν_h^N's is sufficiently rich to express any of its own derivatives. Thus, we will be able to find a recursive decomposition of this derivatives.

Property 7.1 *For all $1 \leqslant j \leqslant n$, for all polynomials $P_1, \cdots, P_\ell$, if ℓ is even,*

$$\frac{\partial}{\partial t_j} \nu_1^N(P_1 \otimes \cdots \otimes P_\ell) = -\nu_1^N(P_1 \otimes \cdots \otimes P_\ell \otimes q_j)$$

$$+ \sum_{r=1}^{\ell} \frac{\partial}{\partial t_j} \mu(P_r) \nu_1^N(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \nu_1^N(P_1 \otimes \cdots \otimes P_\ell) \nu_1^N(q_j)$$

and if ℓ is odd,

$$\frac{\partial}{\partial t_j}\nu_1^N(P_1 \otimes \cdots \otimes P_\ell) = -\nu_2^N(P_1 \otimes \cdots \otimes P_\ell \otimes q_j)$$

$$+ \sum_{r=1}^{\ell} \frac{\partial}{\partial t_j}\mu(P_r)\nu_2^N(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \nu_1^N(P_1 \otimes \cdots \otimes P_\ell)\nu_1^N(q_j).$$

Proof.

We simply need to derivate

$$\frac{1}{Z_V^N}\int \prod_r (\frac{1}{N}\operatorname{tr} -\mu)(P_r)e^{-N\operatorname{tr}V}d\mu^N.$$

In that expression we can either derivate $\frac{1}{Z_V^N}$, the potential $e^{-N\operatorname{tr}V}$ or one of the $\mu(P_r)$, this leads to

$$\frac{\partial}{\partial t_j}E[\prod_r(\frac{1}{N}\operatorname{tr}-\mu)(P_r)] = N^2 E[\prod_r(\frac{1}{N}\operatorname{tr}-\mu)(P_r)(\frac{1}{N}\operatorname{tr}-\mu)(q_j)]$$

$$- E[\prod_r(\frac{1}{N}\operatorname{tr}-\mu)(P_r)]N^2 E[(\frac{1}{N}\operatorname{tr}-\mu)(q_j)]$$

$$+ \sum_r \frac{\partial}{\partial t_j}\mu(P_r)E[\prod_{r'\neq r}(\frac{1}{N}\operatorname{tr}-\mu)(P_{r'})]$$

Where one can notice that we have added in the two first terms of the right hand side the quantity $\mu(q_j)$ but these two modifications cancel each other.

Now multiply by the normalisation N^ℓ to get the equation in the case ℓ even. In the case ℓ odd, if we multiply by $N^{\ell+1}$ we get

$$\frac{\partial}{\partial t_j}\nu_1^N(P_1 \otimes \cdots \otimes P_\ell) = -N^2\nu_1^N(P_1 \otimes \cdots \otimes P_\ell \otimes q_j)$$

$$+ N^2\sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_1^N(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \nu_1^N(P_1 \otimes \cdots \otimes P_\ell)\nu^N(q_j)$$

$$= -\nu_2^N(P_1 \otimes \cdots \otimes P_\ell \otimes q_j)$$

$$+ \sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_2^N(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \nu_1^N(P_1 \otimes \cdots \otimes P_\ell)\nu^N(q_j)$$

$$+ N^2 r_N$$

with by definition of ν_2^N,

$$r_N = -\mathcal{I}_{\frac{l+1}{2}}(P_1 \otimes \cdots \otimes P_\ell \otimes q_j) + \sum_r \frac{\partial}{\partial t_j}\mu(P_r)\mathcal{I}_{\frac{l-1}{2}}(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

Let's have a closer look at this expression, $\mathcal{I}_{\frac{l+1}{2}}(P_1 \otimes \cdots \otimes P_\ell \otimes q_j)$ counts maps with $\frac{l+1}{2}$ handles and such none of the $l+1$ components $P_1, \ldots, P_\ell$, q_j is planar. Since each handle can break the planarity of at most two components, the only way to obtain such a configuration is to form $\frac{l+1}{2}$ couples among these l components and in each of these couples to put a handle between the two components. For example, one can decompose these maps according to the other vertex in the couple of q_j:

$$\mathcal{I}_{\frac{l+1}{2}}(P_1 \otimes \cdots \otimes P_\ell \otimes q_j) = \sum_r \mathcal{I}_1(P_r \otimes q_j)\mathcal{I}_{\frac{l-1}{2}}(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

Now observe that $\mathcal{I}_1(P_r \otimes q_j)$ counts maps of genus 1 such that the component of P_r is linked to the component of q_j, thus it is equivalent to the counting of planar maps with two precribed vertices, one of type P_r and one of type q_j. According to Lemma 4.4, that exactly what count $\frac{\partial}{\partial t_j}\mu(P_r)$. Thus $r_N = 0$ and the Property is proved.

$\square$

With this decomposition we can now show that the derivatives of the ν_h^N's are of order 1.

Lemma 7.2 *For all* $\mathbf{j} = (j_1, \cdots, j_n)$, *for all* $\ell, h \in \mathbb{N}^*$, $\alpha > 0$, *there exists constants* $C, \eta, M_0 > 0$, *such that for all* $\mathbf{t} \in B_{\eta,c}$, $M > M_0$ *and all polynomials* $P_1, \ldots, P_\ell$ *of degree less than* $\alpha N^{\frac{1}{2}}$,

$$|\mathcal{D}_{\mathbf{j}}\nu_h^N(P_1 \otimes \cdots \otimes P_\ell)| \leq C\|P_1\|_M \cdots \|P_\ell\|_M$$

Proof.

The proof is essentially the same than the proof of Lemma 6.2. We just use in addition an induction on $\sum_i j_i$. If $\sum_i j_i = 0$ then we are exactly in the case of the Lemma 6.2. Otherwise, Assume that we have already proved this lemma up to the $\sum_i j_i - 1$-th derivative. To prove it for $\sum_i j_i$, we proceed by induction. First we prove it for ν_1^N, and it is straightforward since we can express the derivatives of ν_1^N as a function of derivatives of lesser degree

according to Property 7.1. Then for $h > 1$ we need to find an induction relation on the $\mathcal{D}_\mathbf{j} \nu_N^h$. We start form Property 5.2:

$$\nu_h^N(\Xi P_1 \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,m} \mu_\mathbf{t}(D_i \bar{P}_1 D_i P_r) \nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{i,m} \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(D_i \bar{P}_1 D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(\partial_i D_i \bar{P}_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

and we take the derivative:

$$\mathcal{D}_\mathbf{j} \nu_h^N(\Xi P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

$$= \sum_{\substack{\mathbf{j}'+\mathbf{j}''=\mathbf{j} \\ \mathbf{j}'' \neq 0}} \binom{\mathbf{j}'}{\mathbf{j}} \mathcal{D}_{\mathbf{j}'} \nu_h^N((\mathcal{D}_{\mathbf{j}''}\Xi)P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

$$+ \sum_{\substack{\mathbf{j}'+\mathbf{j}''=\mathbf{j} \\ \mathbf{j}'' \neq 0}} \sum_{i,m} \binom{\mathbf{j}'}{\mathbf{j}} \mathcal{D}_{\mathbf{j}'} \mu_\mathbf{t}(D_i \bar{P}_1 D_i P_r) \mathcal{D}_{\mathbf{j}''} \nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_{i,m} \mathcal{D}_\mathbf{j} \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(D_i \bar{P}_1 D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

$$+ \sum_i \mathcal{D}_\mathbf{j} \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(\partial_i D_i \bar{P}_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

with

$$\mathcal{D}_{\mathbf{j}''}\Xi P = (I \otimes (\mathcal{D}_{\mathbf{j}''}\mu) + (\mathcal{D}_{\mathbf{j}''}\mu) \otimes I)\partial_i D_i \bar{P} - (\mathcal{D}_{\mathbf{j}''} D_i V)D_i \bar{P}.$$

Thus $\mathcal{D}_\mathbf{j} \nu_h^N$ has a decomposition similar to the one of Property 5.2 and we can use it in similar way we use it in the proof of Lemma 6.2. The rest of the proof is identical to the one of Lemma 6.2 and we only give the main steps.

We define the same Q_n and R_n as in the previous proof to approximate $\Xi^{-1}P$. Then, $\mathcal{D}_\mathbf{j}\nu_h^N(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell)$ is controlled using the previous decomposition, using the fact that $\mathcal{D}_\mathbf{j}\mu$ has a bounded support and that the norm of $(\mathcal{D}_{\mathbf{j}''} D_i V)$ is also bounded. Now write

$$\mathcal{D}_\mathbf{j} \nu_h^N(R_n \otimes P_2 \otimes \cdots \otimes P_\ell) = (\mathcal{D}_\mathbf{j} N^{2(h-1)}\nu_1^N + N^{2(h-1)}\mathcal{D}_\mathbf{j}\mathcal{I}_{\hat{\ell}} + \cdots$$
$$+ \mathcal{D}_\mathbf{j}\mathcal{I}_{\hat{\ell}+h-1})(R_n \otimes P_2 \otimes \cdots \otimes P_\ell)$$

Then for $\mathbf{t}$ sufficiently small these series are convergent and have uniformly bounded support according to Lemma 4.4 and we can use Property 7.1 to control one more time $\mathcal{D}_\mathbf{j}\nu_1^N$.

$\square$

From there we deduce

Property 7.3 *For all ℓ, for all $g \in \mathbb{N}$, for all $\mathbf{j} = (j_1, \cdots, j_n)$ there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all polynomials $P_1, \ldots, P_\ell$,*

$$\mathcal{D}_{\mathbf{j}} \nu^N (P_1 \otimes \cdots \otimes P_\ell) = \mathcal{D}_{\mathbf{j}} \mathcal{I}_{\frac{\ell}{2}} (P_1 \otimes \cdots \otimes P_\ell) + \frac{1}{N^2} \mathcal{D}_{\mathbf{j}} \mathcal{I}_{\frac{\ell}{2}+1} (P_1 \otimes \cdots \otimes P_\ell) +$$

$$\cdots + \frac{1}{N^{2g}} \mathcal{D}_{\mathbf{j}} \mathcal{I}_{\frac{\ell}{2}+g} (P_1 \otimes \cdots \otimes P_\ell) + o(\frac{1}{N^{2g}}).$$

Finally we prove Theorem 1.3,

Theorem 7.4 *For all $\mathbf{j} = (j_1, \cdots, j_n)$, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,*

$$\mathcal{D}_{\mathbf{j}} F^N_{V_{\mathbf{t}}} = \mathcal{D}_{\mathbf{j}} F^0(\mathbf{t}) + \cdots + \mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}}).$$

Besides, $\mathcal{D}_{\mathbf{j}} F^g$ is the generating function for rooted maps of genus g associated with V:

$$\mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) = \sum_{k_1, \cdots, k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{C}_g^{\mathbf{k}+\mathbf{j}}(q_{i_1}, \cdots, q_{i_p})$$

where $\mathcal{C}_g^{k_1, \cdots, k_n}$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

Proof.

The case $\mathbf{j} = 0$ is just Theorem 1.1. Thus we can assume $\mathbf{j} \neq 0$, for example $j_1 \neq 0$. Observe that for all i,

$$\frac{\partial}{\partial t_i} F^N_{V_{\mathbf{t}}} = -E[\hat{\mu}^N(q_i)] = -\mathcal{I}(q_i) - \frac{1}{N^2} \nu^N(q_i).$$

we can use the Property 7.3: there exists $\eta > 0$ such that for $\mathbf{t} \in B_{\eta,c}$,

$$\mathcal{D}_{\mathbf{j}} F^N_{V_{\mathbf{t}}} = -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}} (\mathcal{I}(q_1) + \frac{1}{N^2} \nu^N(q_1))$$

$$= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}} \mathcal{I}(q_1) - \frac{1}{N^2} \mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}} \nu^N(q_1)$$

$$= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}} \mathcal{I}(q_1) - \frac{1}{N^2} \mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}} \mathcal{I}_1(q_1) - \cdots - \frac{1}{N^{2g}} \mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}} \mathcal{I}_g(q_1)$$

$$+ o(\frac{1}{N^{2g}})$$

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Observe now that according to Lemma 4.4

$$\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}_g(q_1) = \sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_g^{\mathbf{k}+\mathbf{j}-\mathbb{1}_{i=1}}(q_1)$$

$$= -\sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}_g^{\mathbf{k}+\mathbf{j}} = -\mathcal{D}_{\mathbf{j}}F_g$$

and by the same method, $\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}(q_1) = -\mathcal{D}_{\mathbf{j}}F_0$. Thus we get,

$$\mathcal{D}_{\mathbf{j}}F_{V_{\mathbf{t}}}^N = \mathcal{D}_{\mathbf{j}}F_0 + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}}F_g + o(\frac{1}{N^{2g}}).$$

$\square$

References

[1] ANÉ, C., BLACHÈRE, S., CHAFAÏ, D., FOUGÈRES, P., GENTIL, I., MALRIEU, F., ROBERTO, C., AND SCHEFFER, G. *Sur les inégalités de Sobolev logarithmiques*, vol. 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.

[2] BERTOLA, M., EYNARD, B., AND HARNAD, J. Duality, biorthogonal polynomials and multi-matrix models. *Comm. Math. Phys. 229*, 1 (2002), 73–120.

[3] DI FRANCESCO P. D., G. P., AND J., Z.-J. 2d gravity and random matrices. *Phys. Rep.*, 254 (1995).

[4] ERCOLANI, N. M., AND MCLAUGHLIN, K. D. T.-R. Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *Int. Math. Res. Not.*, 14 (2003), 755–820.

[5] EYNARD, B., KOKOTOV, A., AND KOROTKIN, D. $1/N^2$-correction to free energy in Hermitian two-matrix model. *Lett. Math. Phys. 71*, 3 (2005), 199–207.

[6] GUIONNET, A. First order asymptotics of matrix integrals ; a rigorous approach towards the understanding of matrix models. *Comm. Math. Phys.* (2003).

[7] GUIONNET, A., AND MAUREL-SEGALA, E. Combinatorial aspects of matrix models. *arviv front.math.ucdavis.edu/math.PR/0503064* (2005).

[8] GUIONNET, A., AND MAUREL-SEGALA, E. Second order asymptotics for matrix models. *arviv front.math.ucdavis.edu/math.PR/0601040* (2006).

[9] TUTTE, W. T. On the enumeration of planar maps. *Bull. Amer. Math. Soc. 74* (1968), 64–74.

[10] ZVONKIN, A. Matrix integrals and map enumeration: an accessible introduction. *Math. Comput. Modelling 26*, 8-10 (1997), 281–304. Combinatorics and physics (Marseilles, 1995).